

Funkcije više promenljivih

Matematika u inženjerskom menadžmentu

Uvod

Funkcija jedne i dve promenljive

Jedna promenljiva: $f : X \rightarrow Y, X \subseteq \mathbb{R}, Y \subseteq \mathbb{R}, f(x) = y$.

Dve promenljive: $f : D \rightarrow Y, D \subseteq \mathbb{R}^2, Y \subseteq \mathbb{R}, f(x, y) = z$.

Primer: $f(x, y) = 2x + 3y$, pa $f(2, 3) = 2 \cdot 2 + 3 \cdot 3 = 13$.

Zadatak 1

Odrediti domene sledećih funkcija

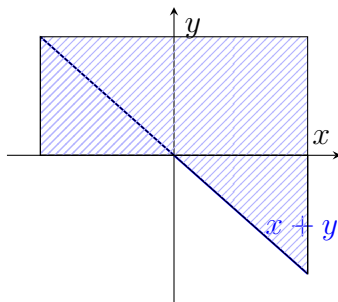
a) $f(x) = \sqrt{x}$ **b)** $f(x, y) = \sqrt{x+y}$ **c)** $f(x, y) = \sqrt{x} + \sqrt{y}$

Rešenje:

a) $x \geq 0 \Rightarrow D_f = [0, +\infty)$.

b) $x + y \geq 0$, tj. prava $x + y = 0$ (tj. $y = -x$) je granica:

$$D_f = \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0\}.$$



c) $x \geq 0$ i $y \geq 0$:

$$D_f = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ i } y \geq 0\}.$$

Parcijalni izvodi prvog reda

Definicija parcijalnih izvoda

Za funkciju $f(x, y)$ u tački $(x_0, y_0) \in D$:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = f'_x(x_0, y_0),$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f'_y(x_0, y_0).$$

Pravilo: Pri računanju $\frac{\partial f}{\partial x}$ tretiramo y kao konstantu i diferenciramo po x .

Pri računanju $\frac{\partial f}{\partial y}$ tretiramo x kao konstantu i diferenciramo po y .

Zadatak 2: Odrediti oba parcijalna izvoda prvog reda

1° $f(x, y) = x^2 + y^2 + xy$

Rešenje:

$$\frac{\partial f}{\partial x} = 2x + 0 + y = 2x + y, \quad \frac{\partial f}{\partial y} = 0 + 2y + x = 2y + x.$$

2° $f(x, y) = x^2y^2 - y^3 + 3x^4 + 5$

Rešenje:

$$\frac{\partial f}{\partial x} = 2xy^2 - 0 + 12x^3 + 0 = 2xy^2 + 12x^3,$$

$$\frac{\partial f}{\partial y} = 2x^2y - 3y^2 + 0 + 0 = 2x^2y - 3y^2.$$

3° $f(x, y) = x^7 + 3y^4 + 3x^2y + 5xy^2 + 7$

Rešenje:

$$\frac{\partial f}{\partial x} = 7x^6 + 0 + 6xy + 5y^2 + 0 = 7x^6 + 6xy + 5y^2,$$

$$\frac{\partial f}{\partial y} = 0 + 12y^3 + 3x^2 + 10xy + 0 = 12y^3 + 3x^2 + 10xy.$$

4° $f(x, y) = xe^y$

Rešenje:

$$\frac{\partial f}{\partial x} = e^y \cdot (x)'_x = e^y, \quad \frac{\partial f}{\partial y} = x \cdot (e^y)'_y = xe^y.$$

5° $f(x, y) = 2^x(xy + y^2)$

Rešenje:

$$\frac{\partial f}{\partial x} = (2^x)'_x(xy+y^2) + 2^x(xy+y^2)'_x = 2^x \ln 2 (xy+y^2) + 2^x \cdot y = 2^x (\ln 2 (xy+y^2) + y),$$

$$\frac{\partial f}{\partial y} = 2^x(xy+y^2)'_y = 2^x(x+2y).$$

6° $z(x, y) = \frac{x+y}{x-y}$

Rešenje:

$$\frac{\partial z}{\partial x} = \frac{(x+y)'_x(x-y) - (x+y)(x-y)'_x}{(x-y)^2} = \frac{(x-y) - (x+y)}{(x-y)^2} = \frac{-2y}{(x-y)^2},$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)'_y(x-y) - (x+y)(x-y)'_y}{(x-y)^2} = \frac{(x-y) + (x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}.$$

7° $z(x, y) = \ln(x-y)$

Rešenje:

$$\frac{\partial z}{\partial x} = \frac{1}{x-y} \cdot (x-y)'_x = \frac{1}{x-y}, \quad \frac{\partial z}{\partial y} = \frac{1}{x-y} \cdot (x-y)'_y = \frac{-1}{x-y}.$$

8° $f(x, y) = \sqrt[3]{x^2y + y^2x} = (x^2y + y^2x)^{1/3}$

Rešenje:

$$\frac{\partial f}{\partial x} = \frac{1}{3}(x^2y + y^2x)^{-2/3} \cdot (2xy + y^2) = \frac{2xy + y^2}{3\sqrt[3]{(x^2y + y^2x)^2}},$$

$$\frac{\partial f}{\partial y} = \frac{1}{3}(x^2y + y^2x)^{-2/3} \cdot (x^2 + 2yx) = \frac{x^2 + 2xy}{3\sqrt[3]{(x^2y + y^2x)^2}}.$$

9° $f(x, y) = (x^2 + 3yx^2 + 3y^3x)e^{x+y}$

Rešenje:

$$\begin{aligned} \frac{\partial f}{\partial x} &= (x^2 + 3yx^2 + 3y^3x)'_x e^{x+y} + (x^2 + 3yx^2 + 3y^3x)e^{x+y} \\ &= (2x + 6xy + 3y^3)e^{x+y} + (x^2 + 3yx^2 + 3y^3x)e^{x+y} \\ &= e^{x+y}(2x + 6xy + 3y^3 + x^2 + 3yx^2 + 3y^3x), \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= (x^2 + 3yx^2 + 3y^3x)'_y e^{x+y} + (x^2 + 3yx^2 + 3y^3x)e^{x+y} \\ &= (3x^2 + 9y^2x)e^{x+y} + (x^2 + 3yx^2 + 3y^3x)e^{x+y} \\ &= e^{x+y}(3x^2 + 9y^2x + x^2 + 3yx^2 + 3y^3x). \end{aligned}$$

10° $f(x, y) = e^{2xy}(x^2 + 3xy + 1)$. Odrediti oba parcijalna izvoda u tački $(1, 1)$.

Rešenje:

$$\begin{aligned}\frac{\partial f}{\partial x} &= (e^{2xy})'_x(x^2 + 3xy + 1) + e^{2xy}(x^2 + 3xy + 1)'_x \\ &= e^{2xy} \cdot 2y \cdot (x^2 + 3xy + 1) + e^{2xy}(2x + 3y) \\ &= e^{2xy}(2y(x^2 + 3xy + 1) + (2x + 3y)) \\ &= e^{2xy}(2x^2y + 6xy^2 + 2y + 2x + 3y) \\ &= e^{2xy}(2x^2y + 6xy^2 + 2x + 5y).\end{aligned}$$

$$\frac{\partial f}{\partial x}(1, 1) = e^2(2 + 6 + 2 + 5) = 15e^2.$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= (e^{2xy})'_y(x^2 + 3xy + 1) + e^{2xy}(x^2 + 3xy + 1)'_y \\ &= e^{2xy} \cdot 2x \cdot (x^2 + 3xy + 1) + e^{2xy} \cdot 3x \\ &= e^{2xy}(2x^3 + 6x^2y + 2x + 3x) = e^{2xy}(2x^3 + 6x^2y + 5x).\end{aligned}$$

$$\frac{\partial f}{\partial y}(1, 1) = e^2(2 + 6 + 5) = 13e^2.$$

11° $\omega(x, y, z) = xyz + x^2y + z^2x$

Rešenje:

$$\frac{\partial \omega}{\partial x} = yz + 2xy + z^2, \quad \frac{\partial \omega}{\partial y} = xz + x^2, \quad \frac{\partial \omega}{\partial z} = xy + 2xz.$$

Parcijalni izvodi drugog reda

Definicija i Schwarzova teorema

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right),$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right).$$

Schwarzova teorema (mešoviti parcijalni izvodi): Ako funkcija $f(x, y)$ ima neprekidne parcijalne izvode $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$ i $\frac{\partial^2 f}{\partial y \partial x}$ na nekom otvorenom skupu $D \subseteq \mathbb{R}^2$, tada važi

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{za svako } (x, y) \in D.$$

Zadatak 3: Odrediti parcijalne izvode drugog reda sledećih funkcija

1° $f(x, y) = x^3 + x^2y + 2xy^2 - 5$

Rešenje:

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy + 2y^2, \quad \frac{\partial f}{\partial y} = x^2 + 4xy.$$
$$\frac{\partial^2 f}{\partial x^2} = 6x + 2y, \quad \frac{\partial^2 f}{\partial y^2} = 4x, \quad \frac{\partial^2 f}{\partial y \partial x} = 2x + 4y, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x + 4y.$$

2° $z(x, y) = x^2 e^{-y}$

Rešenje:

$$\frac{\partial z}{\partial x} = 2xe^{-y}, \quad \frac{\partial z}{\partial y} = -x^2 e^{-y}.$$
$$\frac{\partial^2 z}{\partial x^2} = 2e^{-y}, \quad \frac{\partial^2 z}{\partial y^2} = x^2 e^{-y}, \quad \frac{\partial^2 z}{\partial y \partial x} = -2xe^{-y}, \quad \frac{\partial^2 z}{\partial x \partial y} = -2xe^{-y}.$$

3° $z(x, y) = x^2 + y^2 + xy + x^3y$. Odrediti $\frac{\partial^2 z}{\partial y^2}(1, 2)$ i $\frac{\partial^2 z}{\partial x \partial y}(1, 2)$.

Rešenje:

$$\frac{\partial z}{\partial x} = 2x + y + 3x^2y, \quad \frac{\partial z}{\partial y} = 2y + x + x^3.$$
$$\frac{\partial^2 z}{\partial x^2} = 2 + 6xy, \quad \frac{\partial^2 z}{\partial y^2} = 2, \quad \frac{\partial^2 z}{\partial y \partial x} = 1 + 3x^2, \quad \frac{\partial^2 z}{\partial x \partial y} = 1 + 3x^2.$$
$$\frac{\partial^2 z}{\partial y^2}(1, 2) = 2, \quad \frac{\partial^2 z}{\partial x \partial y}(1, 2) = 1 + 3 \cdot 1 = 4.$$

4° $f(x, y) = \sqrt{x+y}$

Rešenje:

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x+y}}, \quad \frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x+y}}.$$
$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{4\sqrt{(x+y)^3}}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{1}{4\sqrt{(x+y)^3}}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{4\sqrt{(x+y)^3}}.$$

5° $f(x, y) = x^3 + 2xy + x^5y + e^{xy}$

Rešenje:

$$\frac{\partial f}{\partial x} = 3x^2 + 2y + 5x^4y + ye^{xy}, \quad \frac{\partial f}{\partial y} = 2x + x^5 + xe^{xy}.$$
$$\frac{\partial^2 f}{\partial x^2} = 6x + 20x^3y + y^2e^{xy}, \quad \frac{\partial^2 f}{\partial y^2} = x^2e^{xy},$$
$$\frac{\partial^2 f}{\partial y \partial x} = 2 + 5x^4 + e^{xy} + xye^{xy}, \quad \frac{\partial^2 f}{\partial x \partial y} = 2 + 5x^4 + e^{xy} + xye^{xy}.$$

6° $z(x, y) = \ln(x - y)$

Rešenje:

$$\frac{\partial z}{\partial x} = \frac{1}{x-y}, \quad \frac{\partial z}{\partial y} = -\frac{1}{x-y}.$$
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x-y)^2}, \quad \frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x-y)^2},$$
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{(x-y)^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(x-y)^2}.$$

Lokalne ekstremne vrednosti funkcija dve promenljive

Uslov i kriterijum za lokalni ekstrem

Neophodan uslov: Stacionarna tačka $M(x_0, y_0)$ funkcije $f(x, y)$ zadovoljava

$$\frac{\partial f}{\partial x}(M) = 0 \quad \text{ i } \quad \frac{\partial f}{\partial y}(M) = 0.$$

Heseova matrica i diskriminanta:

$$\mathcal{D}(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2.$$

Dovoljan uslov: Neka je M stacionarna tačka.

1° Ako je $\mathcal{D}(M) > 0$, tada f u tački M dostiguje lokalni ekstrem:

- lokalni **minimum** ako je $\frac{\partial^2 f}{\partial x^2}(M) > 0$,
- lokalni **maksimum** ako je $\frac{\partial^2 f}{\partial x^2}(M) < 0$.

2° Ako je $\mathcal{D}(M) < 0$, tada f u tački M **ne dostiguje** ekstremnu vrednost. Tačka M je **sedlasta tačka**.

3° Ako je $\mathcal{D}(M) = 0$, ovaj test ne daje odgovor.

Zadatak 4: Naći lokalne ekstremne vrednosti sledećih funkcija

a) $f(x, y) = x^2 + 6y^2 - 6x + 12y + 10$

Rešenje:

$$\frac{\partial f}{\partial x} = 2x - 6 = 0 \Rightarrow x = 3, \quad \frac{\partial f}{\partial y} = 12y + 12 = 0 \Rightarrow y = -1.$$

Stacionarna tačka: $M(3, -1)$.

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 12, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

$$\mathcal{D}(M) = \begin{vmatrix} 2 & 0 \\ 0 & 12 \end{vmatrix} = 24 > 0, \quad \frac{\partial^2 f}{\partial x^2}(M) = 2 > 0 \Rightarrow M(3, -1) \text{ je lokalni minimum.}$$

b) $f(x, y) = \frac{x^3}{3} + y^2 + 2xy - 15x + 3$

Rešenje:

$$\frac{\partial f}{\partial x} = x^2 + 2y - 15 = 0, \quad \frac{\partial f}{\partial y} = 2y + 2x = 0 \Rightarrow y = -x.$$

Smenom $y = -x$:

$$x^2 - 2x - 15 = 0 \Rightarrow x_{1/2} = \frac{2 \pm 8}{2} : \quad x_1 = 5, y_1 = -5; \quad x_2 = -3, y_2 = 3.$$

Stacionarne tačke: $M_1(5, -5)$ i $M_2(-3, 3)$.

$$\frac{\partial^2 f}{\partial x^2} = 2x, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 2.$$

$$\mathcal{D}(x, y) = \begin{vmatrix} 2x & 2 \\ 2 & 2 \end{vmatrix} = 4x - 4.$$

$$M_1(5, -5): \mathcal{D}(M_1) = 16 > 0, \quad \frac{\partial^2 f}{\partial x^2}(M_1) = 10 > 0 \Rightarrow \text{lokalni minimum.}$$

$$M_2(-3, 3): \mathcal{D}(M_2) = -16 < 0 \Rightarrow M_2 \text{ je sedlasta tačka.}$$

c) $f(x, y) = \frac{1}{3}x^3 + y^3 + 3x^2 + 5x - 3y$

Rešenje:

$$\frac{\partial f}{\partial x} = x^2 + 6x + 5 = 0, \quad \frac{\partial f}{\partial y} = 3y^2 - 3 = 0.$$

$$\text{Iz } x^2 + 6x + 5 = 0: x_{1/2} = \frac{-6 \pm 4}{2}, \text{ pa } x_1 = -1, x_2 = -5.$$

$$\text{Iz } 3y^2 - 3 = 0: y_{1/2} = \pm 1.$$

Stacionarne tačke: $M_1(-1, 1)$, $M_2(-1, -1)$, $M_3(-5, 1)$, $M_4(-5, -1)$.

$$\frac{\partial^2 f}{\partial x^2} = 2x + 6, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

$$\mathcal{D}(x, y) = \begin{vmatrix} 2x + 6 & 0 \\ 0 & 6y \end{vmatrix} = (2x + 6) \cdot 6y.$$

$M_1(-1, 1)$: $\mathcal{D}(M_1) = (4) \cdot 6 = 24 > 0$, $f''_{xx}(M_1) = 4 > 0 \Rightarrow$ **lokalni minimum.**

$M_2(-1, -1)$: $\mathcal{D}(M_2) = (4) \cdot (-6) = -24 < 0 \Rightarrow$ **sedlasta tačka.**

$M_3(-5, 1)$: $\mathcal{D}(M_3) = (-4) \cdot 6 = -24 < 0 \Rightarrow$ **sedlasta tačka.**

$M_4(-5, -1)$: $\mathcal{D}(M_4) = (-4) \cdot (-6) = 24 > 0$, $f''_{xx}(M_4) = -4 < 0 \Rightarrow$ **lokalni maksimum.**

Zadatak 5: Ekonomska primena na dva proizvoda

2. Firma proizvodi dva različita proizvoda. Funkcija prihoda od prvog proizvoda je $P_1 - Q_1^2 - 14Q_1 = 0$, a drugog $P_2 + Q_2 = 6$. Ukupni troškovi: $TC(Q_1, Q_2) = \frac{4}{3}Q_1^3 + 27Q_1 + \frac{1}{2}Q_2^2 + 130$.
Odrediti obim proizvodnje oba proizvoda tako da je dobit maksimalna.

Rešenje:

Iz uslova prihoda:

$$P_1 = Q_1^2 + 14Q_1 \quad (\text{cena prvog proizvoda}), \quad P_2 = 6 - Q_2 \quad (\text{cena drugog proizvoda}).$$

$$TR(Q_1, Q_2) = Q_1 \cdot P_1 + Q_2 \cdot P_2 = Q_1(Q_1^2 + 14Q_1) + Q_2(6 - Q_2) = Q_1^3 + 14Q_1^2 - Q_2^2 + 6Q_2.$$

$$\Pi(Q_1, Q_2) = TR - TC =$$

$$\begin{aligned} &= Q_1^3 + 14Q_1^2 - Q_2^2 + 6Q_2 - \left(\frac{4}{3}Q_1^3 + 27Q_1 + \frac{1}{2}Q_2^2 + 130\right) = \\ &= Q_1^3 + 14Q_1^2 - Q_2^2 + 6Q_2 - \frac{4}{3}Q_1^3 - 27Q_1 - \frac{1}{2}Q_2^2 - 130 = \\ &= -\frac{1}{3}Q_1^3 + 14Q_1^2 - 27Q_1 - \frac{3}{2}Q_2^2 + 6Q_2 - 130. \end{aligned}$$

$$\frac{\partial \Pi}{\partial Q_1} = -Q_1^2 + 28Q_1 - 27 = 0 \quad | \cdot (-1)$$

$$Q_1^2 - 28Q_1 + 27 = 0$$

$$Q_1^{(1/2)} = \frac{28 \pm \sqrt{784 - 108}}{2} = \frac{28 \pm 26}{2} \Rightarrow Q_1 = 1 \text{ ili } Q_1 = 27,$$

$$\frac{\partial \Pi}{\partial Q_2} = -3Q_2 + 6 = 0 \Rightarrow Q_2 = 2.$$

Stacionarne tačke: $M_1(1, 2)$ i $M_2(27, 2)$.

$$\frac{\partial^2 \Pi}{\partial Q_1^2} = -2Q_1 + 28, \quad \frac{\partial^2 \Pi}{\partial Q_2^2} = -3, \quad \frac{\partial^2 \Pi}{\partial Q_1 \partial Q_2} = 0.$$

$$\mathcal{D}(Q_1, Q_2) = \begin{vmatrix} -2Q_1 + 28 & 0 \\ 0 & -3 \end{vmatrix} = -3(-2Q_1 + 28) = 6Q_1 - 84.$$

$M_1(1, 2)$: $\mathcal{D}(M_1) = 6 - 84 = -78 < 0 \Rightarrow$ sedlasta tačka.

$M_2(27, 2)$: $\mathcal{D}(M_2) = 6 \cdot 27 - 84 = 78 > 0$, $\frac{\partial^2 \Pi}{\partial Q_1^2}(M_2) = -2 \cdot 27 + 28 = -26 < 0 \Rightarrow$

lokalni maksimum.

$$\boxed{Q_1 = 27, \quad Q_2 = 2.}$$

Zadatak 6: Ekonomska primena na maksimum dobiti firme

3. Funkcija dobiti firme koja proizvodi dva različita proizvoda x i y pri ceni je: $\Pi(x, y) = -3x^2 + 5xy - 4y^2 - 4x + 103y - 118$. Odrediti obim proizvodnje oba proizvoda tako da je dobit maksimalna.

Rešenje:

$$\frac{\partial \Pi}{\partial x} = -6x + 5y - 4 = 0, \quad \frac{\partial \Pi}{\partial y} = 5x - 8y + 103 = 0.$$

Sistem:

$$\begin{cases} -6x + 5y = 4 \\ 5x - 8y = -103 \end{cases} \begin{array}{l} \cdot 5 \\ \cdot 6 \end{array} \Rightarrow \begin{cases} -30x + 25y = 20 \\ 30x - 48y = -618 \end{cases} \Rightarrow -23y = -598 \Rightarrow y = 26.$$

$$-6x + 5 \cdot 26 = 4 \Rightarrow -6x = 4 - 130 = -126 \Rightarrow x = 21.$$

Stacionarna tačka: $M(21, 26)$.

$$\frac{\partial^2 \Pi}{\partial x^2} = -6, \quad \frac{\partial^2 \Pi}{\partial y^2} = -8, \quad \frac{\partial^2 \Pi}{\partial x \partial y} = 5.$$

$$\mathcal{D}(M) = \begin{vmatrix} -6 & 5 \\ 5 & -8 \end{vmatrix} = 48 - 25 = 23 > 0, \quad \frac{\partial^2 \Pi}{\partial x^2}(M) = -6 < 0.$$

$\Rightarrow$ Funkcija dostiguje **lokalni maksimum** u tački $M(21, 26)$.

$$\boxed{x = 21, \quad y = 26.}$$