Тригонометрија

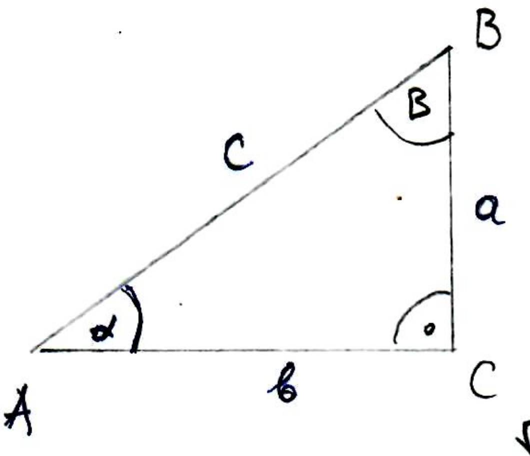
Углови:

$2\pi = 360^\circ$, $\pi = 180^\circ$, $\frac{\pi}{2} = 90^\circ$, $\frac{\pi}{3} = 60^\circ$, $\frac{\pi}{4} = 45^\circ$

$\frac{\pi}{6} = 30^\circ$ .... $\frac{3\pi}{2} = \pi + \frac{\pi}{2} = 180^\circ + 90^\circ = 270^\circ$

$135^\circ = 90^\circ + 45^\circ = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

Правоугли троугао

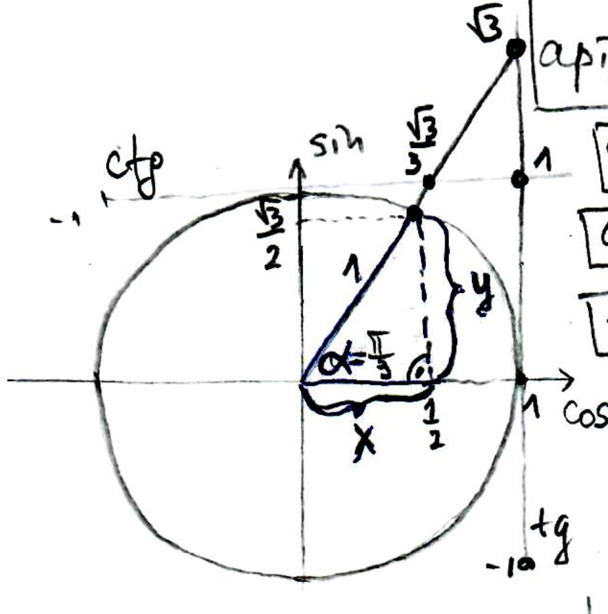


$\boxed{\sin\alpha = \frac{a}{c}}$, $\boxed{\cos\alpha = \frac{b}{c}}$

$\boxed{tg\,\alpha} = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}} = \boxed{\frac{a}{b}}$

$\boxed{ctg\,\alpha} = \frac{\cos\alpha}{\sin\alpha} = \frac{\frac{b}{c}}{\frac{a}{c}} = \boxed{\frac{b}{a}}$

sin, cos, tg, ctg без аргумената не означавају ништа!

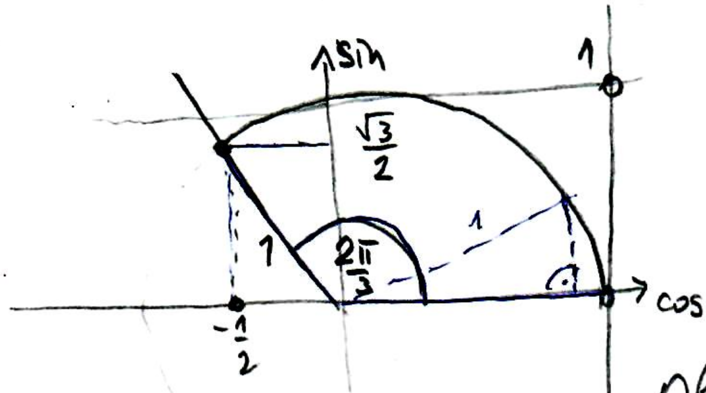


$\boxed{\sin\alpha} = \sin\frac{\pi}{3} = \frac{y}{1} = y = \frac{\sqrt{3}}{2}$

$\boxed{\cos\alpha} = \cos\frac{\pi}{3} = \frac{x}{1} = x = \frac{1}{2}$

$\boxed{tg\,\alpha} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\boxed{ctg\,\alpha} = \frac{x}{y} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



$\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6} = 90^\circ + 30^\circ = 120^\circ$

$\sin\frac{2\pi}{3} = \sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \boxed{\frac{\sqrt{3}}{2}}$

$\cos\frac{2\pi}{3} = \cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = \boxed{-\frac{1}{2}}$

Ове вредности су добијене формулама свођења!

Коришћењем адиционе ф-ле за збир:

$$\sin\left(\frac{2\pi}{3}\right)=\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\sin\frac{\pi}{2}\cos\frac{\pi}{6}+\cos\frac{\pi}{2}\sin\frac{\pi}{6}=1\cdot\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2\pi}{3}\right)=\cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right)=\cos\frac{\pi}{2}\cos\frac{\pi}{6}-\sin\frac{\pi}{2}\sin\frac{\pi}{6}=-1\cdot\sin\frac{\pi}{6}=-\frac{1}{2}$$

угао α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$tg\,\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	не постоји
$ctg\,\alpha$	не постоји	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

$\sin(-\alpha)=-\sin\alpha$ – НЕПАРНОСТ

$\cos(-\alpha)=\cos\alpha$ – ПАРНОСТ

$\sin(-2\alpha)\neq-2\sin\alpha$

$\sin 3\alpha\neq 3\sin\alpha$!

$\cos 2\alpha\neq 2\cos\alpha$

$$\boxed{\sin^2\alpha+\cos^2\alpha=1}$$

ОСНОВНА ТРИГОНОМЕТРИЈСКА ЈЕДНАКОСТ (ИДЕНТИТЕТ)

$\sin x^2=\sin(x^2)$

$\sin^2 x=(\sin x)^2$!

29 Наћи сва решења једначине

$$\cos 2x - 5\sin x - 3 = 0$$

Решење:

Косинус двоструког угла
$$\boxed{\cos 2x = \cos^2 x - \sin^2 x}$$

$$\cos 2x - 5\sin x - 3 = 0 \Rightarrow \cos^2 x - \sin^2 x - 5\sin x - 3 = 0$$

Из основне тригонометријске једнакости $\boxed{\sin^2 x + \cos^2 x = 1}$ је $\cos^2 x = 1 - \sin^2 x$.

$$\Rightarrow (1-\sin^2 x) - \sin^2 x - 5\sin x - 3 = 0$$

$$\Rightarrow -2\sin^2 x - 5\sin x - 2 = 0 \quad / \cdot (-1)$$

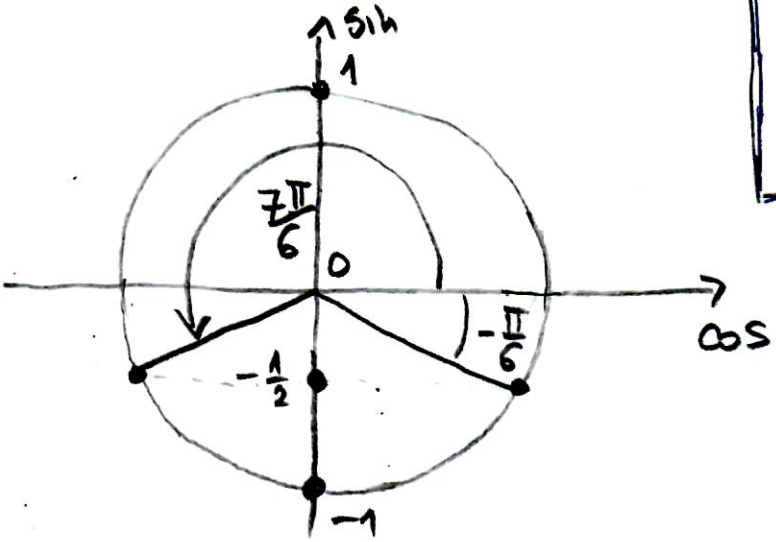
$$\Rightarrow 2\sin^2 x + 5\sin x + 2 = 0.$$

Увођењем смене $\boxed{\sin x = t}$ једначина постаје:

$$2t^2 + 5t + 2 = 0$$

$$t_{1,2} = \frac{-5 \pm \sqrt{5^2 - 4\cdot 2\cdot 2}}{2\cdot 2} = \frac{-5 \pm \sqrt{9}}{4} = \frac{-5 \pm 3}{4} \rightarrow \boxed{t_1 = -\frac{1}{2}} \quad \boxed{t_2 = -2}$$

1° $t_1 = \sin x = -\frac{1}{2}$



$$\boxed{\begin{array}{l} x = \frac{7\pi}{6} + 2k\pi, \ k \in \mathbb{Z} \\ x = -\frac{\pi}{6} + 2n\pi, \ n \in \mathbb{Z} \end{array}}$$

2° $t_2 = \sin x = -2$ – немогуће

Образложење: $-1 \le \sin x \le 1$

22 Наћи сва решења једначине
$$\cos 2x - 4\sin^2 x\cos^2 x = 1$$

Решење:
(I начин)

$\boxed{\begin{array}{l}\text{синус двоструког угла}\\ \sin 2x = 2\sin x\cos x\end{array}} \Rightarrow \sin^2 2x = 4\sin^2 x\cos^2 x$

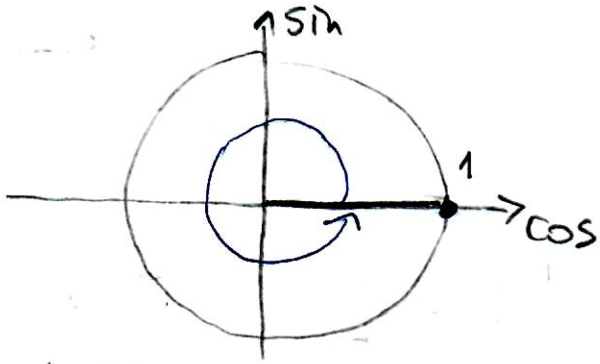
Заменом у једначину је:
$\cos 2x - \sin^2 2x = 1$. Из основне тригонометријске једнакости узимајући угао $2x$ је $\sin^2 2x + \cos^2 2x = 1$
$$\Downarrow$$
$$\boxed{\sin^2 2x = 1 - \cos^2 2x}$$

Једначина постаје:
$\cos 2x - (1-\cos^2 2x) = 1 \Rightarrow \cos^2 2x + \cos 2x - 2 = 0$

Увођењем смене $\boxed{\cos 2x = t}$ је
$t^2 + t - 2 = 0$

$$t_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1} = \frac{-1\pm 3}{2} \begin{array}{l}\nearrow \boxed{t_1 = 1}\\ \searrow \boxed{t_2 = -2}\end{array}$$

1° $(t_1 = 1)$
$\cos 2x = 1$



$2x = 2k\pi, \quad k\in\mathbb{Z}$
$$\boxed{x = k\pi, \quad k\in\mathbb{Z}}$$

2° $(t_2 = -2)$

$\cos 2x = -2$ — немогуће

Решење: (II начин)

$$\boxed{\cos 2x = \cos^2x - \sin^2x} = (\underbrace{1-\sin^2x}_{\text{основна тр. једн.}}) - \sin^2x$$

$$\Rightarrow \boxed{\cos 2x = 1 - 2\sin^2x}$$

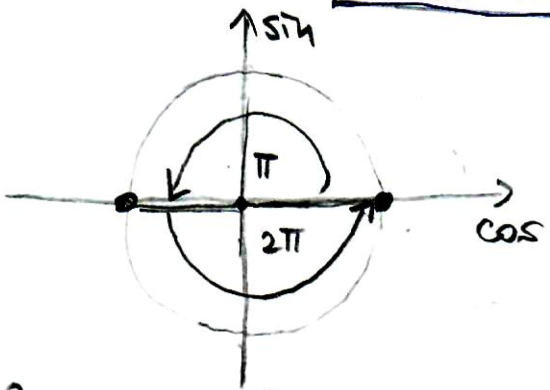
Једначина постаје:

$$\underbrace{\cancel{1}^{0} - 2\sin^2x}_{\cos 2x} - 4\sin^2x\cos^2x = \cancel{1}^{0} \quad /:(-2)$$

$$\Rightarrow \sin^2x + 2\sin^2x\cdot\cos^2x = 0 \Rightarrow \sin^2x\,(1+2\cos^2x) = 0$$

1° $\sin^2x = 0$ $\qquad$ 2° $1+2\cos^2x = 0$

1° $\sin^2x = 0 \Rightarrow \boxed{\sin x = 0}$ $\Rightarrow \boxed{x = k\pi\,,\ k\in\mathbb{Z}}$



2° $1+2\cos^2x = 0 \Rightarrow \cos^2x = -\frac{1}{2}$ Немогуће!

Квадрат не може бити негативан!

10 Наћи сва решења ј-не

$$\sin^2x - 3\cos^2x + 2\sin 2x = 1$$

Решење:

$\boxed{\sin 2x = 2\sin x\cdot\cos x}$ синус двоструког угла

$\boxed{\sin^2x + \cos^2x = 1}$ основна тр. једнакост

$$\cancel{\sin^2x} - 3\cos^2x + \underbrace{4\sin x\cos x}_{2\sin 2x} = \underbrace{\cancel{\sin^2x} + \cos^2x}_{1} \quad /-\cos^2x$$

$$-4\cos^2x + 4\sin x\cos x = 0 \quad /:(-4)$$

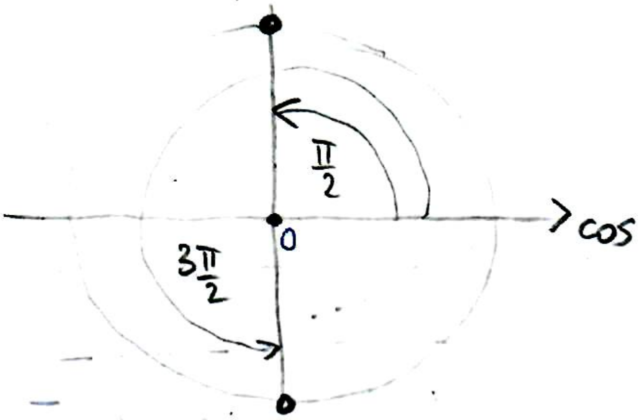
$$\cos^2x - \sin x\cos x = 0 \Rightarrow$$

$$\cos x \cdot (\cos x - \sin x) = 0$$

1° $\cos x = 0$ 2° $\cos x - \sin x = 0$ тј. $\sin x = \cos x$

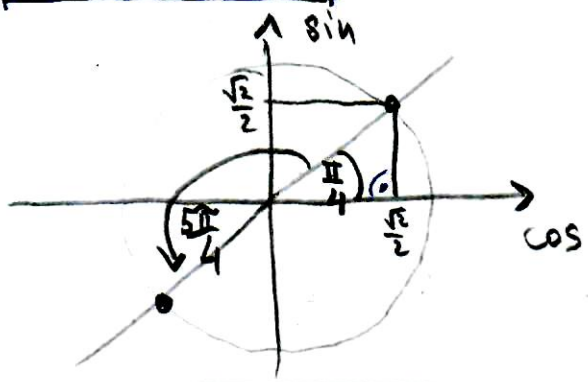
1° $\cos x = 0$

$$\boxed{x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}}$$



2° $\sin x = \cos x$

$$\boxed{x = \frac{\pi}{4} + n\pi, \quad n \in \mathbb{Z}}$$



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23 Наћи сва решења ј-не

$$\cos 2x + \cos x = \sin 2x + \sin x$$

Решење: Користимо ф-ле за трансформацију збира у производ

$$\boxed{\cos\alpha + \cos\beta = 2 \cdot \cos\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \cos \underset{\alpha}{2x} + \cos \underset{\beta}{x} = 2 \cdot \cos\frac{3x}{2} \cdot \cos\frac{x}{2}$$

$$\boxed{\sin\alpha + \sin\beta = 2 \cdot \sin\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \sin 2x + \sin x = 2 \cdot \sin\frac{3x}{2} \cdot \cos\frac{x}{2}$$

Једначина постаје:

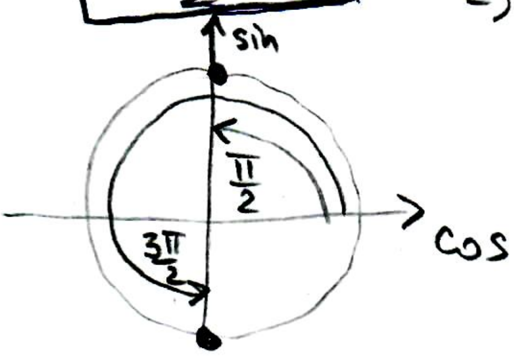
$$2\cos\frac{3x}{2}\cos\frac{x}{2} = 2\sin\frac{3x}{2}\cdot\cos\frac{x}{2} \quad /:2$$

$$\Rightarrow \cos\frac{3x}{2}\cos\frac{x}{2} - \sin\frac{3x}{2}\cos\frac{x}{2} = 0 \Rightarrow$$

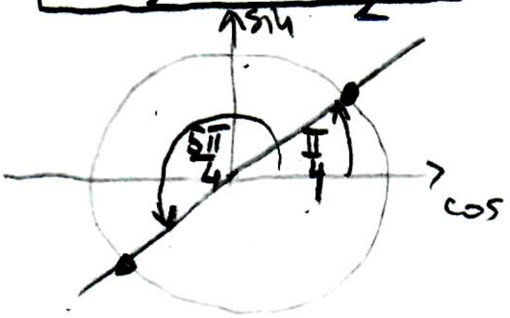
$$\Rightarrow \underbrace{\cos\frac{x}{2}}\left(\underbrace{\cos\frac{3x}{2} - \sin\frac{3x}{2}}\right) = 0$$

1° $\cos\frac{x}{2} = 0$ 2° $\cos\frac{3x}{2} - \sin\frac{3x}{2} = 0$ тј. $\cos\frac{3x}{2} = \sin\frac{3x}{2}$

1° $\cos\frac{x}{2} = 0 \Rightarrow \frac{x}{2} = \frac{\pi}{2} + k\pi \quad /\cdot 2 \Rightarrow \boxed{x = \pi + 2k\pi,\ k\in\mathbb{Z}}$



2° $\cos\frac{3x}{2} = \sin\frac{3x}{2} \Rightarrow \frac{3x}{2} = \frac{\pi}{4} + n\pi \quad /\cdot\frac{2}{3}$



$$\Rightarrow \boxed{x = \frac{\pi}{6} + \frac{2n}{3}\pi \quad n\in\mathbb{Z}}$$

(20) Наћи сва решења ј-не

$$\sin^2 x + \sin^2 2x = 1$$

Решење:

$$\sin^2 x+\sin^2 2x = 1 \Rightarrow \sin^2 2x = \underbrace{1-\sin^2 x = \cos^2 x}_{\text{основни идентитет}}$$

$$\Rightarrow \sin^2 2x = \cos^2 x \Rightarrow \boxed{4\sin^2 x\cos^2 x = \cos^2 x}$$

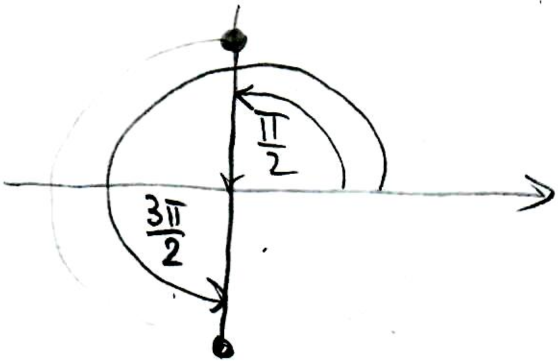
синус двоструког угла

$$\sin 2x = 2\sin x\cos x \Rightarrow \sin^2 2x = 4\sin^2 x\cos^2 x$$

$$\Rightarrow 4\sin^2 x\cos^2 x - \cos^2 x = 0 \Rightarrow \cos^2 x\,(4\sin^2 x-1)=0$$

1° $\cos^2 x = 0$ $\qquad$ 2° $4\sin^2 x - 1 = 0$

1° $\cos^2 x = 0 \Rightarrow \cos x = 0$

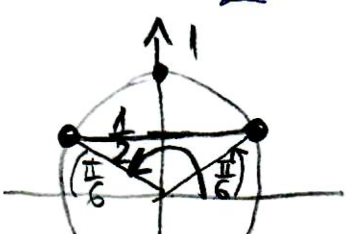


$$\boxed{x = \frac{\pi}{2} + k\pi, \quad k\in\mathbb{Z}}$$

2° $4\sin^2 x - 1 = 0 \Rightarrow \sin^2 x = \frac{1}{4} \Rightarrow \sin x = \pm\sqrt{\frac{1}{4}}$

a) $\boxed{x_1 = \frac{1}{2}}$ $\qquad$ б) $\boxed{x_2 = -\frac{1}{2}}$

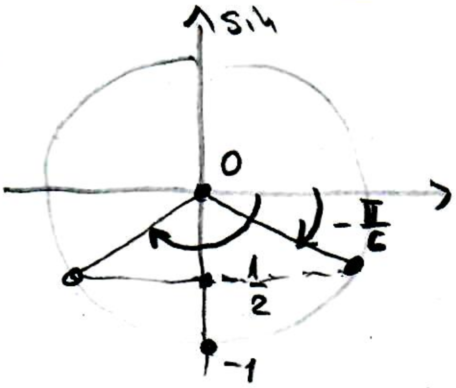
a) $\sin x = \frac{1}{2} \Rightarrow \boxed{x = \frac{\pi}{6} + 2n\pi, n\in\mathbb{Z}}$



$$\boxed{x = \frac{5\pi}{6} + 2m\pi, m\in\mathbb{Z}}$$

б) $\sin x = -\frac{1}{2} \Rightarrow$ $\boxed{x = -\frac{\pi}{6} + 2r\pi, \ r \in \mathbb{Z}}$

$\boxed{x = -\frac{5\pi}{6} + 2s\pi, \ s \in \mathbb{Z}}$



21 Решити тригонометријску једначину

$$\sin\left(\frac{\pi}{4} - x\right) \cdot \sin\left(\frac{\pi}{4} + x\right) = -\frac{1}{2}$$

ТРАНСФОРМАЦИЈА
ПРОИЗВОДА У ЗБИР

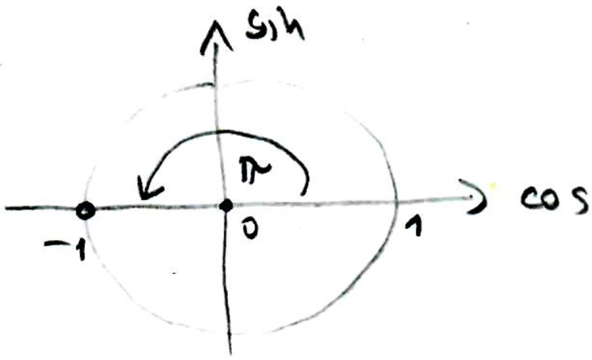
$$\boxed{\sin\alpha \cdot \sin\beta = \frac{1}{2}\left(\cos(\alpha - \beta) - \cos(\alpha + \beta)\right)}$$

$$\sin\underbrace{\left(\frac{\pi}{4} - x\right)}_{\alpha} \cdot \sin\underbrace{\left(\frac{\pi}{4} + x\right)}_{\beta} = \frac{1}{2}\left(\cos\left(\frac{\pi}{4} - x - \left(\frac{\pi}{4} + x\right)\right) - \cos\left(\frac{\pi}{4} - x + \frac{\pi}{4} + x\right)\right)$$

$$= \frac{1}{2}\left(\cos(-2x) - \cos\frac{\pi}{2}\right) = \frac{1}{2}\cos(-2x) = \frac{1}{2}\cos 2x$$

Једначина постаје:

$$\frac{1}{2}\cos 2x = -\frac{1}{2} \quad / \cdot 2 \Rightarrow \quad \cos 2x = -1$$



$$2x = \pi + 2k\pi \quad / : 2$$

$$\boxed{x = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}}$$

(12) Решити неједначину:

$$-2\cos^2 x + \sin x + 1 < 0$$

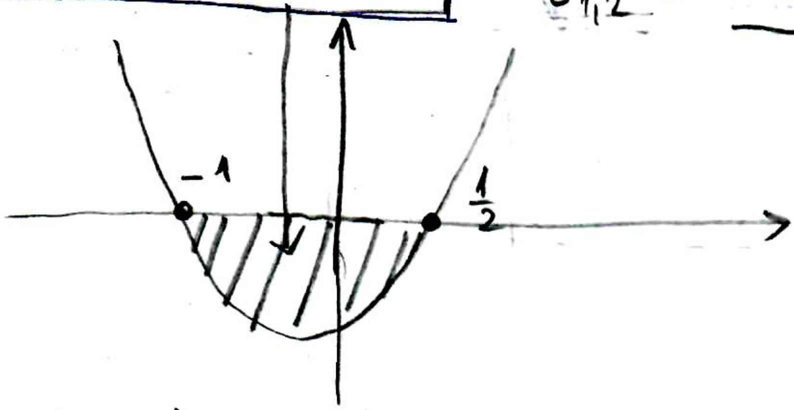
$\boxed{\begin{array}{c}\cos^2 x = 1 - \sin^2 x \\ \text{ОСНОВНА ТРИГ.} \\ \text{ЈЕДНАКОСТ}\end{array}}$ $\Rightarrow (-2)(1-\sin^2 x) + \sin x + 1 < 0$

$\Rightarrow 2\sin^2 x + \sin x - 1 < 0$ $\boxed{\sin x = t}$

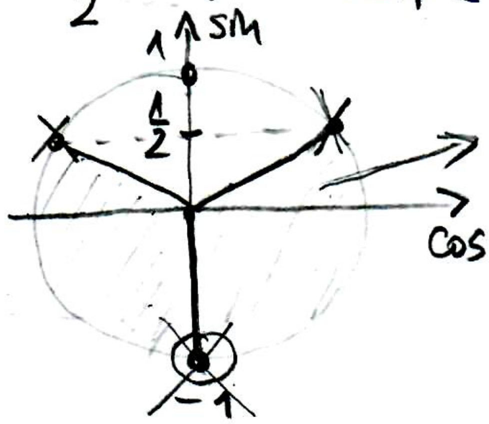
$$\boxed{2t^2 + t - 1 < 0}$$

$$t_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4\cdot 2(-1)}}{2\cdot 2} = \frac{-1 \pm 3}{4}$$

$\boxed{t_1 = -1}$ $\boxed{t_2 = \frac{1}{2}}$



$$-1 < t < \frac{1}{2} \Rightarrow -1 < \sin x < \frac{1}{2}$$



$$\boxed{-\frac{\pi}{2} + 2k\pi < x < \frac{\pi}{6} + 2k\pi,\ k \in Z}$$

$$\boxed{\frac{5\pi}{6} + 2n\pi < x < \frac{3\pi}{2} + 2n\pi,\ n \in Z}$$

1. Ако је $\alpha+\beta+\gamma=180^\circ$ доказати да је:

$$tg\alpha+tg\beta+tg\gamma=tg\alpha\cdot tg\beta\cdot tg\gamma.$$

Решење: $\alpha+\beta+\gamma=\pi \Rightarrow \gamma=\pi-(\alpha+\beta)$

тангенс разлике

$$\boxed{tg(s-t)=\frac{tgs-tgt}{1+tgs\cdot tgt}}$$

$$tg\pi=\frac{\sin\pi}{\cos\pi}=\frac{0}{1}=0$$

$$tg(\gamma)=tg(\underbrace{\pi}_{s}-\underbrace{(\alpha+\beta)}_{t})=\frac{\overset{0}{tg\pi}-tg(\alpha+\beta)}{1+\overset{0}{tg\pi}\cdot tg(\alpha+\beta)}=-tg(\alpha+\beta)$$

$$tg\alpha+tg\beta+tg\gamma=tg\alpha+tg\beta-tg(\alpha+\beta)$$

$$=tg\alpha+tg\beta-\frac{tg\alpha+tg\beta}{1-tg\alpha\cdot tg\beta}$$

$$\boxed{tg(s+t)=\frac{tgs+tgt}{1-tgs\cdot tgt}}$$

тангенс збира

$$=(tg\alpha+tg\beta)\left(1-\frac{1}{1-tg\alpha\cdot tg\beta}\right)$$

$$=(tg\alpha+tg\beta)\frac{\cancel{1}-tg\alpha\cdot tg\beta-\cancel{1}}{1-tg\alpha\cdot tg\beta}=(tg\alpha+tg\beta)\left(-\frac{tg\alpha\cdot tg\beta}{1-tg\alpha\cdot tg\beta}\right)$$

$$=tg\alpha\cdot tg\beta\cdot\left(-\frac{tg\alpha+tg\beta}{1-tg\alpha\cdot tg\beta}\right)=tg\alpha\cdot tg\beta\cdot(\underbrace{-tg(\alpha+\beta)}_{tg\gamma})$$

$$=tg\alpha\cdot tg\beta\cdot tg\gamma$$

⑥ Доказати идентитет

$$\operatorname{tg}\left(\alpha+\frac{\pi}{4}\right)=\frac{1+\sin 2\alpha}{\cos 2\alpha}$$

Решење

$$\boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$
трансформација синуса збира

$$\boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$
трансформација косинуса збира

$$\boxed{\sin\frac{\pi}{4}=\frac{\sqrt2}{2}\qquad \cos\frac{\pi}{4}=\frac{\sqrt2}{2}}$$

$$\sin\left(\alpha+\frac{\pi}{4}\right)=\sin\alpha\cos\frac{\pi}{4}+\cos\alpha\sin\frac{\pi}{4}=\frac{\sqrt2}{2}(\sin\alpha+\cos\alpha)$$

$$\cos\left(\alpha+\frac{\pi}{4}\right)=\cos\alpha\cos\frac{\pi}{4}-\sin\alpha\sin\frac{\pi}{4}=\frac{\sqrt2}{2}(\cos\alpha-\sin\alpha)$$

Сада је:

$$\operatorname{tg}\left(\alpha+\frac{\pi}{4}\right)=\frac{\sin\left(\alpha+\frac{\pi}{4}\right)}{\cos\left(\alpha+\frac{\pi}{4}\right)}=\frac{\cancel{\frac{\sqrt2}{2}}(\sin\alpha+\cos\alpha)}{\cancel{\frac{\sqrt2}{2}}(\cos\alpha-\sin\alpha)}\cdot 1$$

$$=\frac{\sin\alpha+\cos\alpha}{\cos\alpha-\sin\alpha}\cdot\underbrace{\frac{\cos\alpha+\sin\alpha}{\cos\alpha+\sin\alpha}}_{1}=\frac{(\sin\alpha+\cos\alpha)^2}{\cos^2\alpha-\sin^2\alpha}$$

$$=\frac{\overbrace{\sin^2\alpha+2\sin\alpha\cos\alpha+\cos^2\alpha}^{1}}{\cos^2\alpha-\sin^2\alpha}=$$

$$\boxed{\sin 2\alpha=2\sin\alpha\cos\alpha}$$
синус двоструког угла

$$=\frac{1+2\sin\alpha\cos\alpha}{\cos^2\alpha-\sin^2\alpha}=$$

$$\boxed{\cos 2\alpha=\cos^2\alpha-\sin^2\alpha}$$
косинус двоструког угла

$$=\boxed{\frac{1+\sin 2\alpha}{\cos 2\alpha}}$$

7 Доказати идентитет

$$\frac{\cos\alpha}{ctg^2\frac{\alpha}{2}-tg^2\frac{\alpha}{2}}=\frac{1}{4}\sin^2\alpha$$

Решење:

$$\boxed{\sin^2\frac{t}{2}=\frac{1-\cos t}{2}} \qquad \boxed{\cos^2\frac{t}{2}=\frac{1+\cos t}{2}}$$

$$tg^2\frac{\alpha}{2}=\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}=\frac{\frac{1-\cos\alpha}{2}}{\frac{1+\cos\alpha}{2}}=\frac{1-\cos\alpha}{1+\cos\alpha}$$

Слично је:

$$ctg^2\frac{\alpha}{2}=\frac{\cos^2\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}}=\frac{\frac{1+\cos\alpha}{2}}{\frac{1-\cos\alpha}{2}}=\frac{1+\cos\alpha}{1-\cos\alpha}$$

$$\frac{\cos\alpha}{ctg^2\frac{\alpha}{2}-tg^2\frac{\alpha}{2}}=\frac{\cos\alpha}{\frac{1+\cos\alpha}{1-\cos\alpha}-\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{\frac{\cos\alpha}{1}}{\frac{(1+\cos\alpha)^2-(1-\cos\alpha)^2}{(1-\cos\alpha)(1+\cos\alpha)}}$$

$$=\frac{\frac{\cos\alpha}{1}}{\frac{\cancel{1}+2\cos\alpha+\cancel{\cos^2\alpha}-(\cancel{1}-2\cos\alpha+\cancel{\cos^2\alpha})}{1-\cos^2\alpha}}=\frac{\cancel{\cos\alpha}(1-\cos^2\alpha)}{4\cancel{\cos\alpha}}$$

$$=\boxed{\frac{1}{4}\sin^2\alpha}$$